

A variant of the Bombieri-Vinogradov theorem with explicit constants

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Abstract

In this paper we improve the result of [1] with getting $(\log x)^{\frac{7}{2}}$ instead of $(\log x)^{\frac{9}{2}}$. In particular we obtain a better version of Vaughan's inequality by applying the explicit variant of an inequality connected to the Möbius function from [2].

1 Introduction

For integer number a and $q \geq 1$, let

$$\psi(x; q, a) = \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \Lambda(n),$$

where $\Lambda(n)$ is the von Mangoldt function. The Bombieri-Vinogradov theorem is an estimate for the error terms in the prime number theorem for arithmetic progressions averaged over all $q \leq x^{1/2}$.

Theorem. (Bombieri-Vinogradov) *Let A be a given positive number and $Q \leq x^{1/2}(\log x)^{-B}$ where $B = B(A)$, then*

$$\sum_{q \leq Q} \max_{2 \leq y \leq x} \max_{\substack{a \\ (a, q) = 1}} \left| \psi(y, q, a) - \frac{y}{\phi(q)} \right| \ll_A \frac{x}{(\log x)^A}.$$

The implied constant in this theorem is not effective, since we have to take care of characters, associated with those q that have small prime factors. The main result of this paper is

Theorem 1. (Bombieri-Vinogradov theorem with explicit constants) *Let $x \geq 4$, $1 \leq Q_1 \leq Q \leq x^{\frac{1}{2}}$. Let also $l(q)$ denote the least prime divisor of q . Define $F(x, Q, Q_1)$ by*

$$F(x, Q, Q_1) = \frac{14x}{Q_1} + 4x^{\frac{1}{2}}Q + 15x^{\frac{2}{3}}Q^{\frac{1}{2}} + 4x^{\frac{5}{6}} \log \frac{Q}{Q_1}.$$

Then

$$\sum_{\substack{q \leq Q \\ l(q) > Q_1}} \max_{2 \leq y \leq x} \max_{\substack{a \\ (a, q) = 1}} \left| \psi(y; q, a) - \frac{\psi(y)}{\phi(q)} \right| < c_1 F(x, Q, Q_1) (\log x)^{\frac{7}{2}},$$

where

$$\begin{aligned} c_1 &= \frac{5}{4} E_0 c_0 + 1 = 42.140461 \dots, \\ E_0 &= \prod_p \left(1 + \frac{1}{p(p-1)} \right) = 1.943596 \dots, \\ c_0 &= (2A_0)^{\frac{1}{2}} \frac{2^5}{3^{\frac{3}{2}} \pi (\log 2)^2} \left(2 + \frac{\log(\log 2)}{\log \frac{4}{3}} \right) = 16.93375 \dots, \\ A_0 &= \max_{x > 0} \left(\frac{\psi(x)}{x} \right) = \frac{\psi(113)}{113} = 1.03883 \dots \end{aligned}$$

Previously the best result obtained by these methods in the literature is due to Akbary, Hambrook (see [1, Theorem 1.3]), where they proved that under assumptions of Theorem 1 we have.

$$\sum_{\substack{q \leq Q \\ l(q) > Q_1}} \max_{2 \leq y \leq x} \max_{\substack{a \\ (a, q) = 1}} \left| \psi(y; q, a) - \frac{\psi(y)}{\phi(q)} \right| < c_1 F(x, Q, Q_1) (\log x)^{\frac{9}{2}},$$

where $F(x, Q, Q_1)$ is defined by

$$F(x, Q, Q_1) = \frac{4x}{Q_1} + 4x^{\frac{1}{2}}Q + 18x^{\frac{2}{3}}Q^{\frac{1}{2}} + 5x^{\frac{5}{6}} \log \frac{eQ}{Q_1}.$$

Here we reduce this power to $(\log x)^{\frac{7}{2}}$ by applying an explicit version for an upper bound for

$$b_k = \sum_{\substack{d \leq V \\ d|k}} \mu(d),$$

where $\mu(d)$ is Mobius function, V is a given number. This version can be found in [2], namely we have

Lemma 1. (Helfgott, [2, (6.9), (6.10)]) *For V large enough we have*

$$\sum_{k \leq Y} |b_k|^2 = Y(L + O^*(C)) + O^*(V^2), \text{ where } C = 0.000023, L = 0.440729$$

and $O^*(x)$ means that it is less in absolute value than x .

This Lemma is a variant of the sum considered in [3], where it is shown that

$$\sum_{d_1, d_2 \leq Y} \frac{\mu(d_1)\mu(d_2)}{\gcd(d_1, d_2)}$$

tends to a positive constant as $Y \rightarrow \infty$. It is also suggested without proving that L can be about 0.440729.

Notice, that by sharpening the inequality in Lemma 1 we will not be able to reduce the power of $\log x$, since the upper bound is optimal there, so by these methods the power $\frac{7}{2}$ is the best possible. Going further seems to be a hard problem which involves among simpler things a very careful analysis of the logarithmic mean of Möbius function twisted by a Dirichlet character.

Remark. Let $Q = \frac{x^{\frac{1}{2}}}{(\log x)^B}$, where $B > \frac{7}{2}$. Then Theorem 1 gives us the following bound

$$\sum_{\substack{q \leq Q \\ l(q) > Q_1}} \max_{2 \leq y \leq x} \max_{\substack{a \\ (a, q) = 1}} \left| \psi(y; q, a) - \frac{\psi(y)}{\phi(q)} \right| < c_1 \left(\frac{14x}{Q_1} (\log x)^{\frac{7}{2}} + 19x (\log x)^{\frac{7}{2}-B} \right).$$

Remark. It would be very good for applications to get $(\log x)^2$ in Theorem 1, however it seems impossible to get by present methods.

Remark 1. Define

$$\pi(x) = \sum_{p \leq x} 1 \quad \text{and} \quad \pi(x; q, a) = \sum_{\substack{p \leq x \\ p \equiv a \pmod{q}}} 1.$$

Then Theorem 1 under the same assumptions can be also formulated for $\pi(x)$, $\pi(x; q, a)$:

$$\sum_{\substack{q \leq Q \\ l(q) > Q_1}} \max_{2 \leq y \leq x} \max_{\substack{a \\ (a, q) = 1}} \left| \pi(y; q, a) - \frac{\pi(y)}{\phi(q)} \right| < c_2 F(x, Q, Q_1) (\log x)^{\frac{7}{2}},$$

where $c_2 = 1 + \frac{2c_1}{\log 2}$.

Proof of the remark is exactly the same as in [1], we just have to change the power of \log .

The key tool for the proof of Theorem 1 is Vaughan's identity, which we have to get in an explicit version for our goal. Define

$$\psi(y, \chi) = \sum_{n \leq y} \Lambda(n) \chi(n),$$

the twisted summatory function for the von Mangoldt function Λ and a Dirichlet character χ modulo q . One of two main results of this paper is

Proposition 1. (Vaughan's inequality in an explicit form) For $x \geq 4$

$$\sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi(q)}^* \max_{y \leq x} |\psi(y, \chi)| < c_0 (7x + 2Q^2 x^{\frac{1}{2}} + 5Q^{\frac{3}{2}} x^{\frac{2}{3}} + 4Qx^{\frac{5}{6}}) (\log x)^{\frac{5}{2}},$$

where Q is any positive real number and $\sum_{\chi(q)}^*$ means a sum over all primitive characters $\chi \pmod{q}$.

The goal is to get an explicit version of $f(x, Q)$ by applying an improved version of Pólya-Vinogradov inequality (see [4]), that will reduce the coefficients of $f(x, Q)$ and then we can apply Lemma 1.

2 Proof of Proposition 1

Fix arbitrary real numbers $Q > 0$ and $x \geq 4$. In this section, we shall establish Proposition 1, which is the main ingredient in the proof of Theorem 1. Here we follow the ideas of [1] and applying the results from [2]. The main tool in the proof is the large sieve inequality (see, for example [5, p.561])

$$\sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi(q)}^* \left| \sum_{m=m_0+1}^{m_0+M} a_m \chi(m) \right|^2 \leq (M + Q^2) \sum_{m=m_0+1}^{m_0+M} |a_m|^2, \quad (1)$$

from which it follows (see [1, Lemma 6.1]) that

$$\begin{aligned} \sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi(q)}^* \max_y \left| \sum_{m=m_0}^M \sum_{\substack{n=n_0 \\ mn \leq y}}^N a_m b_n \chi(mn) \right| \leq \\ c_3 (M' + Q^2)^{\frac{1}{2}} (N' + Q^2)^{\frac{1}{2}} \left(\sum_{m=m_0}^M |a_m|^2 \right)^{\frac{1}{2}} \left(\sum_{n=n_0}^N |b_n|^2 \right)^{\frac{1}{2}} L(M, N), \end{aligned} \quad (2)$$

where $c_3 = 2.64\dots$, $L(M, N) = \log(2MN)$ and $M' = M - m_0 + 1$, $N' = N - n_0 + 1$ are the number of terms in the sums over m and n respectively. Here the a_m , b_n are arbitrary complex numbers.

2.1 Sieving and Vaughan's identity

We reduce to the case $2 \leq Q \leq x^{1/2}$. If $Q < 1$, then the sum on the left-hand side of (1) is empty and we are done. Next, $1 \leq Q < 2$ then only the $q = 1$ term exists and we have

$$\sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi(q)}^* \max_{y \leq x} |\psi(y, \chi)| = \max_{y \leq x} \left| \sum_{n \leq y} \Lambda(n) \right| = \psi(x) \leq A_0 x, \quad (3)$$

which is better than the theorem. Finally, if $Q > x^{1/2}$, Theorem 1 follows from (2) with $M = m_0 = n_0 = 1$, $N = \lfloor x \rfloor$, $a_m = 1$, $b_n = \Lambda(n)$ by the estimate

$$\sum_{n \leq x} \Lambda(n)^2 \leq \psi(x) \log x \leq A_0 x \log x.$$

From now on we assume $2 \leq Q \leq x^{1/2}$. Notice that the fact that we can restrict ourselves to the range $2 \leq Q \leq x^{1/2}$ allows us to apply Lemma 1 (otherwise it would make less sense, since the main term in Lemma 1 would be smaller than O^* -term). As in [1] we will use Vaughan's identity (see also [6])

$$\Lambda(n) = \lambda_1(n) + \lambda_2(n) + \lambda_3(n) + \lambda_4(n),$$

where

$$\begin{aligned}\lambda_1(n) &= \begin{cases} \Lambda(n), & \text{if } n \leq U, \\ 0, & \text{if } n > U, \end{cases} & \lambda_2(n) &= \sum_{\substack{hd=n \\ d \leq V}} \mu(d) \log h, \\ \lambda_3(n) &= - \sum_{\substack{mdr=n \\ m \leq U, d \leq V}} \Lambda(m) \mu(d), & \lambda_4(n) &= - \sum_{\substack{mk=n \\ m > U, k > V}} \Lambda(m) \sum_{\substack{d|k \\ d \leq V}} \mu(d).\end{aligned}$$

Assume $y \leq x$, $q \leq Q$, and χ is a character mod q . We use the above decomposition to write

$$\psi(y, \chi) = S_1 + S_2 + S_3 + S_4,$$

where

$$S_i = \sum_{n \leq y} \lambda_i(n) \chi(n).$$

Let U, V be non-negative functions of x and Q to be set later and denote the contributions to our main sum by

$$\mathcal{S}_i = \sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi(q)}^* \max_{y \leq x} |S_i|.$$

Easily we obtain

$$\sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi(q)}^* \max_{y \leq x} |\psi(y, \chi)| \leq \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4.$$

The heart of the proof of Theorem 1.3 in [1] are the following estimates:

Lemma. (Akbari, Hambrook, [1, Section 7]) *We have*

$$\begin{aligned}\mathcal{S}_1 &\leq A_0 U Q^2, \quad \mathcal{S}_2 < \left(x + Q^{\frac{5}{2}} V\right) (\log x V)^2, \quad \mathcal{S}_3 < \mathcal{S}_3' + \mathcal{S}_3'', \\ \mathcal{S}_3' &< (x + Q^{\frac{5}{2}} U) (\log x U)^2, \\ \mathcal{S}_3'' &< \frac{c_3}{\log 2} \left(x + Q x^{\frac{1}{2}} U^{\frac{1}{2}} V^{\frac{1}{2}} + 2^{\frac{1}{2}} Q x U^{-\frac{1}{2}} + Q^2 x^{\frac{1}{2}}\right) (\log 2UV)^2 (\log 4x), \\ \mathcal{S}_4 &< \frac{2^{\frac{3}{2}} A_1^{\frac{1}{2}} c_3}{\log 2} (x + Q x V^{-\frac{1}{2}} + 2^{\frac{1}{2}} Q x U^{-\frac{1}{2}} + Q^2 x^{\frac{1}{2}}) \left(\log \frac{2x}{V}\right)^{\frac{3}{2}} (\log e^3 V) (\log 4x).\end{aligned}$$

where c_3 as in (??).

We estimate \mathcal{S}_4 contribution with the use of Lemma 1. Writing S_4 as a dyadic sum we have

$$S_4 = - \sum_{\substack{M=2^\alpha \\ \frac{1}{2}U < M \leq x/V}} \sum_{\substack{U < m \leq x/V \\ M < m \leq 2M}} \sum_{\substack{V < k \leq x/M \\ mk \leq y}} \Lambda(m) \left(\sum_{\substack{d|k \\ d \leq V}} \mu(d) \right) \chi(mk).$$

Using the triangle inequality

$$\mathcal{S}_4 \leq \sum_{\substack{M=2^\alpha \\ \frac{1}{2}U < M \leq x/V}} \sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi(q)}^* \max_{y \leq x} \left| \sum_{\substack{U < m \leq x/V \\ M < m \leq 2M}} \sum_{\substack{V < k \leq x/M \\ mk \leq y}} a_m b_k \chi(mk) \right|,$$

where $a_m = \Lambda(m)$, and, as it was defined in the introduction $b_k = \sum_{d|k, d \leq V} \mu(d)$. Now apply the large sieve inequality (2) to get

$$\mathcal{S}_4 \leq c_3 \sum_{\substack{M=2^\alpha \\ \frac{1}{2}U < M \leq x/V}} (M' + Q^2)^{\frac{1}{2}} (K' + Q^2)^{\frac{1}{2}} \sigma_1(M)^{\frac{1}{2}} \sigma_2(M)^{\frac{1}{2}} L(M)$$

where

$$\sigma_1(M) = \sum_{V < k \leq x/M} |b_k|^2, \quad \sigma_2(M) = \sum_{\substack{U < m \leq x/V \\ M < m \leq 2M}} |a_m|^2,$$

and

$$L(M) = \log \left(\frac{2x}{M} \min \left(\frac{x}{V}, 2M \right) \right) \leq \log 4x,$$

where M' and K' denote the number of terms in the sums over m and k , respectively. From the definition of M' and N' we conclude

$$M' = \min \left(2M, \frac{x}{V} \right) - \max(M + 1, U + 1) \leq M,$$

$$K' = \frac{x}{M} - (V + 1) + 1 \leq \frac{x}{M}.$$

By Chebyshev estimate we have an upper bound

$$\sigma_2(M) \leq \sum_{m \leq 2M} \Lambda(m)^2 \leq \psi(2M) \log 2M \leq 2A_0 M \log 2M.$$

Thus by Cauchy inequality

$$\mathcal{S}_4 \leq c_3(\log 4x) \sum_{\substack{M=2^\alpha \\ \frac{1}{2}U < M \leq x/V}} (M + Q^2)^{\frac{1}{2}} \left(\frac{x}{M} + Q^2 \right)^{\frac{1}{2}} (2A_0 M \log 2M)^{\frac{1}{2}} \sigma_1(M)^{\frac{1}{2}}. \quad (4)$$

Further

$$M(M + Q^2) \left(\frac{x}{M} + Q^2 \right) = Mx + Q^2x + M^2Q^2 + MQ^4$$

and

$$(\log 2M)^{\frac{1}{2}} \leq \left(\log \frac{2x}{V} \right)^{\frac{1}{2}}.$$

Using Lemma 1 we get

$$(\sigma_1(M))^{\frac{1}{2}} \leq \frac{x}{M}(L + C) - V(L + C) + 2V^2,$$

that implies

$$\mathcal{S}_4 \leq c_3(2A_0)^{\frac{1}{2}}(x + 2^{\frac{1}{2}}Q^{\frac{1}{2}}xU^{-\frac{1}{2}} + QxV^{-\frac{1}{2}} + Q^2x^{\frac{1}{2}})(\log 4x) \left(\log \frac{2x}{V}\right)^{\frac{1}{2}} \sum_{\substack{M=2^\alpha \\ \frac{1}{2}U < M \leq x/V}} 1.$$

Since

$$\sum_{\substack{M=2^\alpha \\ \frac{1}{2}U < M \leq x/V}} 1 \leq \frac{\log \frac{2x}{V}}{\log 2},$$

then

$$\mathcal{S}_4 \leq \frac{c_3}{\log 2}(2A_0)^{\frac{1}{2}}(x + 2^{\frac{1}{2}}Q^{\frac{1}{2}}xU^{-\frac{1}{2}} + QxV^{-\frac{1}{2}} + Q^2x^{\frac{1}{2}})(\log 4x) \left(\log \frac{2x}{V}\right)^{\frac{3}{2}}.$$

Combining it with results of Lemma 2.1 we get

$$\mathcal{S} = \sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi(q)}^* \max_{y \leq x} |\psi(y, \chi)| \leq c_4 \text{Rat}(x, Q, U, V) \text{Log}(x, V, U), \quad (5)$$

where

$$\begin{aligned} c_4 &= \max \left\{ A_0, \frac{c_3}{\log 2}, \frac{c_3}{\log 2}(2A_0)^{\frac{1}{2}} \right\} = \frac{c_3}{\log 2}(2A_0)^{\frac{1}{2}}, \\ \text{Rat}(x, Q, U, V) &= 4x + 2Q^2x^{\frac{1}{2}} + UQ^2 + Q^{\frac{5}{2}}(U + V) + \\ &\quad + 2^{\frac{1}{2}}Q^{\frac{1}{2}}xU^{-\frac{1}{2}} + 2^{\frac{1}{2}}QxU^{-\frac{1}{2}} + Qx^{\frac{1}{2}}U^{\frac{1}{2}}V^{\frac{1}{2}} + QxV^{-\frac{1}{2}}, \\ \text{Log}(x, V, U) &= \max \left\{ (\log xV)^2, (\log xU)^2, (\log 2UV)^2 \log 4x, \left(\log \frac{2x}{V}\right)^{\frac{3}{2}} \log 4x \right\}, \end{aligned}$$

Now let's specify U and V . If $x^{\frac{1}{3}} \leq Q \leq x^{\frac{1}{2}}$, then $U = V = x^{\frac{2}{3}}Q^{-1}$. Then putting that into previous expression we get for the factor

$$\begin{aligned} \text{Rat}_1(x, Q) &= 4x + 2Q^2x^{\frac{1}{2}} + Qx^{\frac{2}{3}}(1 + 2^{\frac{1}{2}}) + Q^{\frac{3}{2}}x^{\frac{2}{3}}(2 + 2^{\frac{1}{2}} + 1) + x^{\frac{7}{6}} \leq \\ &\leq 4x + 2Q^2x^{\frac{1}{2}} + 2Qx^{\frac{5}{6}} + Q^{\frac{3}{2}}x^{\frac{2}{3}}(2 + 2^{\frac{1}{2}} + 1). \end{aligned}$$

where we used the fact that $x^{\frac{7}{6}} \leq Qx^{\frac{5}{6}}$ and $Qx^{\frac{2}{3}} \leq Qx^{\frac{5}{6}}$. Working in the same manner with Log and keeping in mind the condition $x \geq 4$ we find that

$$\text{Log}_1(x, V, U) \leq \left(\frac{4}{3} \log x\right)^{\frac{3}{2}} 2 \log x = \frac{2^4}{3^{\frac{3}{2}}} (\log x)^{\frac{5}{2}}.$$

If $Q \leq x^{\frac{1}{3}}$, we let $U = V = x^{\frac{1}{3}}$ and get

$$\begin{aligned} \text{Rat}_2(x, Q) &= 4x + 2Q^2x^{\frac{1}{2}} + Q^2x^{\frac{1}{3}} + 2Q^{\frac{5}{2}}x^{\frac{1}{3}} + 2^{\frac{1}{2}}Q^{\frac{1}{2}}x^{\frac{5}{6}} + Qx^{\frac{5}{6}}(2^{\frac{1}{2}} + 2) \leq \\ &\leq x(5 + 2^{\frac{1}{2}}) + 2Q^2x^{\frac{1}{2}} + 2Q^{\frac{3}{2}}x^{\frac{2}{3}} + Qx^{\frac{5}{6}}(2^{\frac{1}{2}} + 2), \end{aligned}$$

where we used $Q^2 x^{\frac{1}{3}} \leq x$, $Q^{\frac{5}{2}} x^{\frac{1}{3}} \leq Q^{\frac{3}{2}} x^{\frac{2}{3}}$ and $Q^{\frac{1}{2}} x^{\frac{5}{6}} \leq x$.

Similarly we get for

$$\text{Log}_2(x, V, U) \leq 2 \left(\frac{7}{6} \right)^{\frac{3}{2}} (\log x)^{\frac{5}{2}}.$$

Finally, we have in (5)

$$\mathcal{S} \leq c_4 \frac{2^4}{3^{\frac{3}{2}}} (7x + 2Q^2 x^{\frac{1}{2}} + 5Q^{\frac{3}{2}} x^{\frac{2}{3}} + 4Qx^{\frac{5}{6}}) (\log x)^{\frac{5}{2}},$$

as demanded.

3 Proof of Theorem 1

Let $y \geq 2, (a, q) = 1$. By orthogonality of characters modulo q , we have

$$\psi(y; q, a) = \frac{1}{\phi(q)} \sum_{\chi} \bar{\chi}(a) \psi(y, \chi).$$

Define $\psi'(y, \chi) = \psi(y, \chi)$ if $\chi \neq \chi_0$ and $\psi'(y, \chi) = \psi(y, \chi) - \psi(y)$ otherwise, χ_0 is the principal character mod q . Then

$$\psi(y, q, a) - \frac{\psi(y)}{\phi(q)} = \frac{1}{\phi(q)} \sum_{\chi} \bar{\chi}(a) \psi'(y, \chi).$$

For a character $\chi \pmod{q}$, we let χ^* be the primitive character modulo q^* inducing χ . Follow the way of [1] we obtain

$$\psi'(y, \chi^*) - \psi'(y, \chi) = \psi(y, \chi^*) - \psi(y, \chi) = \sum_{p^k \leq y} (\log p) (\chi^*(p^k) - \chi(p^k)).$$

If $p|q$ then $(p^k, q^*) = 1$, and hence $\chi^*(p^k) = \chi(p^k)$. If $p \nmid q$ then $\chi(p^k) = 0$. Therefore

$$|\psi'(y, \chi^*) - \psi'(y, \chi)| \leq \sum_{\substack{p^k \leq y \\ p \nmid q}} (\log p) \leq (\log y) \sum_{p|q} 1 \leq (\log qy)^2.$$

Denote the quantity we want to estimate as

$$\mathcal{M} = \sum_{\substack{q \leq Q \\ l(q) > Q_1}} \max_{2 \leq y \leq x} \max_{\substack{a \\ (a, q) = 1}} \left| \psi(y; q, a) - \frac{\psi(y)}{\phi(q)} \right|.$$

Since

$$\left| \psi(y, q, a) - \frac{\psi(y)}{\phi(q)} \right| \leq \frac{1}{\phi(q)} \sum_{\chi} |\psi'(y, \chi)| \leq (\log qy)^2 + \frac{1}{\phi(q)} \sum_{\chi} |\psi'(y, \chi^*)|,$$

then

$$\mathcal{M} \leq Q(\log Qx)^2 + \sum_{\substack{q \leq Q \\ l(q) > Q_1}} \frac{1}{\phi(q)} \sum_{\chi} \max_{2 \leq y \leq x} |\psi'(y, \chi^*)|.$$

We have to take care just of the second term in the inequality above, since the first one is smaller than the desired bound. It remains to prove

$$\mathcal{N} = \sum_{\substack{q \leq Q \\ l(q) > Q_1}} \frac{1}{\phi(q)} \sum_{\chi} \max_{2 \leq y \leq x} |\psi'(y, \chi^*)| \leq (c_1 - 1)F(x, Q, Q_1)(\log x)^4,$$

where $F(x, Q, Q_1)$ is the function from Theorem 1. A primitive character $\chi^* \bmod q^*$ induces characters of moduli dq^* and $\psi'(y, \chi^*) = 0$ for χ principal, we observe

$$\mathcal{N} = \sum_{\substack{q \leq Q \\ l(q) > Q_1}} \frac{1}{\phi(q)} \sum_{\substack{q^* | q \\ q^* \neq 1}}^* \sum_{\chi(q^*)} \max_{2 \leq y \leq x} |\psi'(y, \chi)| \leq \sum_{\substack{q^* \leq Q \\ l(q^*) > Q_1}} \sum_{\chi(q^*)}^* \max_{2 \leq y \leq x} |\psi'(y, \chi)| \sum_{k \leq \frac{Q}{q^*}} \frac{1}{\phi(kq^*)}.$$

As it was noted in [1] for $x > 0$

$$\sum_{k \leq x} \frac{1}{\phi(k)} \leq E_0 \log(ex)$$

and as $q^* \leq Q \leq x^{1/2}$, $\phi(k)\phi(q^*) \leq \phi(kq^*)$ and $x \geq 4$, we have

$$\sum_{k \leq \frac{Q}{q^*}} \frac{1}{\phi(kq^*)} < \frac{5E_0}{4\phi(q^*)} \log x.$$

For $q > 1$ and χ primitive character $(\bmod q)$, we know that χ is non-principal and $\psi(y, \chi) = \psi'(y, \chi)$. Since we assumed $Q_1 \geq 1$ then we can replace $\psi'(y, \chi)$ by $\psi(y, \chi)$ inside the internal sum for \mathcal{N} . Combining it with an expression for \mathcal{N} we get

$$\mathcal{N} \leq \frac{5E_0}{4}(\log x) \sum_{\substack{q \leq Q \\ l(q) > Q_1}} \frac{1}{\phi(q)} \sum_{\chi(q)}^* \max_{2 \leq y \leq x} |\psi(y, \chi)| = \mathcal{R}.$$

Thus it remains to show that

$$\mathcal{R} \leq \frac{4(c_1 - 1)}{5E_0} F(x, Q, Q_1)(\log x)^{\frac{5}{2}}.$$

Let

$$\mathcal{R}(q) = \frac{q}{\phi(q)} \sum_{\chi(q)}^* \max_{2 \leq y \leq x} |\psi(y, \chi)|.$$

Partial summation gives us

$$\sum_{Q_1 < q \leq Q} \frac{1}{\phi(q)} \sum_{\chi(q)}^* \max_{2 \leq y \leq x} |\psi(y, \chi)| = \frac{1}{Q} \sum_{q \leq Q} \mathcal{R}(q) - \frac{1}{Q_1} \sum_{q \leq Q_1} \mathcal{R}(q) + \int_{Q_1}^Q \left(\sum_{q \leq t} \mathcal{R}(q) \right) \frac{dt}{t}.$$

Now we apply Theorem 1

$$\sum_{q \leq Q} \mathcal{R}(q) < c_0 f(x, Q) (\log x)^{\frac{5}{2}},$$

where $f(x, Q) = 7x + 2Q^2 x^{\frac{1}{2}} + 5Q^{\frac{3}{2}} x^{\frac{2}{3}} + 4Qx^{\frac{5}{6}}$. Then

$$\sum_{Q_1 < q \leq Q} \frac{1}{\phi(q)} \sum_{\chi(q)}^* \max_{2 \leq y \leq x} |\psi(y, \chi)| < c_0 \left(\Delta_f(Q, Q_1) + \int_{Q_1}^Q f(x, t) \frac{dt}{t} \right) (\log x)^{\frac{5}{2}},$$

where

$$\Delta_f(Q, Q_1) = \frac{f(x, Q)}{Q} - \frac{f(x, Q_1)}{Q_1} \leq \frac{7x}{Q_1} + 2x^{\frac{1}{2}}Q + 5x^{\frac{2}{3}}Q^{\frac{1}{2}}.$$

Calculating the integrals gives us

$$\int_{Q_1}^Q f(x, t) \frac{dt}{t} < \frac{7x}{Q_1} + 2x^{\frac{1}{2}}Q + 10x^{\frac{2}{3}}Q^{\frac{1}{2}} + 4x^{\frac{5}{6}} \log \frac{Q}{Q_1}.$$

Finally

$$\mathcal{N} \leq \frac{4(c_1 - 1)}{5E_0} \left(\frac{14x}{Q_1} + 4x^{\frac{1}{2}}Q + 15x^{\frac{2}{3}}Q^{\frac{1}{2}} + 4x^{\frac{5}{6}} \log \frac{Q}{Q_1} \right) (\log x)^{\frac{5}{2}}.$$

3.1 Proof of Remark 1

Define two functions

$$\pi_1(y) = \sum_{2 \leq n \leq y} \frac{\Lambda(n)}{\log n} \quad \text{and} \quad \pi_1(y; q, a) = \sum_{\substack{2 \leq n \leq y \\ n \equiv a \pmod{q}}} \frac{\Lambda(n)}{\log n}.$$

Since

$$\pi_1(y; q, a) - \pi(y; q, a) = \sum_{2 \leq k \leq \frac{\log y}{\log 2}} \sum_{\substack{p^k \leq y \\ p^k \equiv a \pmod{q}}} \frac{1}{k} \leq \sum_{2 \leq k \leq \frac{\log y}{\log 2}} \frac{\pi(y^{\frac{1}{k}})}{2} < 2y^{\frac{1}{2}},$$

where we used the fact that for $x > 1$ (see for example [1, Lemma 3.1])

$$\pi(x) < 1.25506 \frac{x}{\log x}.$$

Similarly, $\pi_1(y) - \pi(y) < 2y^{\frac{1}{2}}$. Thus by partial summation we obtain the bound

$$\begin{aligned} \left| \pi_1(y; q, a) - \frac{\pi_1(y)}{\phi(q)} \right| &= \left| \frac{\psi(y; q, a) - \psi(y)/\phi(q)}{\log y} - \int_2^y \frac{\psi(t; q, a) - \psi(t)/\phi(q)}{t \log^2 t} dt \right| \\ &\leq \frac{1}{\log 2} \left| \psi(y; q, a) - \frac{\psi(y)}{\phi(q)} \right| + \max_{2 \leq t \leq y} \left| \psi(t; q, a) - \frac{\psi(t)}{\phi(q)} \right| \left(\frac{1}{\log 2} - \frac{1}{\log y} \right). \end{aligned}$$

We have

$$\begin{aligned}
& \sum_{\substack{q \leq Q \\ l(q) > Q_1}} \max_{2 \leq y \leq x} \max_{\substack{a \\ (a, q) = 1}} \left| \pi(y; q, a) - \frac{\pi(y)}{\phi(q)} \right| \\
& \leq \frac{2}{\log 2} \sum_{\substack{q \leq Q \\ l(q) > Q_1}} \max_{2 \leq y \leq x} \max_{a, (a, q) = 1} \left| \psi(y, q, a) - \frac{\psi(y)}{\phi(q)} \right| + 2x^{\frac{1}{2}} \sum_{\substack{q \leq Q \\ l(q) > Q_1}} \left(1 + \frac{1}{\phi(q)} \right) \\
& < \frac{2c_1}{\log 2} F(x, Q, Q_1) (\log x)^{\frac{7}{2}} + 2x^{\frac{1}{2}} \sum_{\substack{q \leq Q \\ l(q) > Q_1}} \left(1 + \frac{1}{\phi(q)} \right),
\end{aligned}$$

where we used Theorem 1 to estimate the first summand. For $x \geq 4$

$$2x^{\frac{1}{2}} \sum_{\substack{q \leq Q \\ l(q) > Q_1}} \left(1 + \frac{1}{\phi(q)} \right) < \frac{2c_1}{\log 2} F(x, Q, Q_1) (\log x)^{\frac{7}{2}}.$$

and we are done.

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